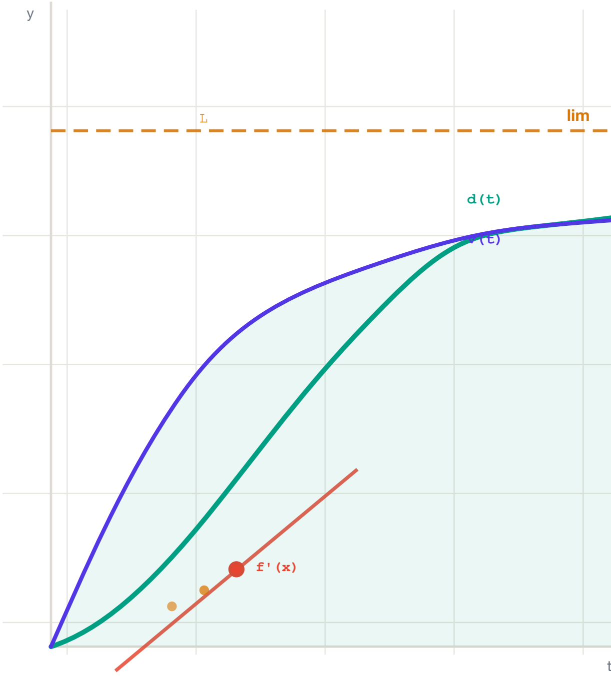
CALCULUS FOR ATHLETES

Limits, Derivatives & Integrals explained through sprinting, throwing, scoring, and winning.

01 · Limits | 02 · Derivatives | 03 · Integrals



REAL ATHLETES · REAL MATH · REAL INSIGHT

CH. 01

Limits

How close can you get to perfect? Why world records can't fall forever — and what calculus says about the boundary.

CH. 02

Derivatives

Your speed at a single instant. The tangent line, instantaneous rate of change, and why a 0.01 s gap in a race matters.

CH. 03

Integrals

Adding up the tiny moments. How your GPS watch computes total distance — and the math behind cumulative stats.

INTRODUCTION

Why Calculus?

Because every world record, every perfect throw, and every split-second decision in sports is governed by mathematics.

When Usain Bolt crossed the finish line in 9.58 seconds in Berlin 2009, thousands of calculations happened in that race. His **peak velocity** at the 65-meter mark, the **rate at which his speed changed** during the acceleration phase, and the **total distance his legs generated force** — all of these are calculus questions.

Modern sports are saturated with data. GPS vests track a soccer player's position 10 times per second. Pitch-tracking systems capture a baseball's trajectory in three dimensions. Wearables measure a cyclist's power output in real time. Behind all of this technology — and the numbers it produces — are three fundamental ideas from calculus.

Limits

What value does a function *approach*? Why can't world records keep falling forever? What is the theoretical boundary of human performance?

Derivatives

How fast is something *changing right now*? What is your instantaneous speed at the 50-meter mark? Where does your shot reach its peak?

Integrals

How much has *accumulated* over time? Total distance covered, total energy expended, total work done by a force — these are all integrals.

> *We didn't used to know things like peak velocity, acceleration profiles, ground-contact time. Now we can measure all of it. And behind every one of those metrics is calculus.*
>
> — SPORTS SCIENCE PERSPECTIVE ON MODERN ATHLETICS

How to Use This Guide

Each chapter introduces one big idea, connects it to a real sport, and gives you the tools to work with it mathematically. You don't need to be a math expert — you just need to bring what you already know about sports: how things speed up, slow down, and accumulate.

- **Sprinting & track:** acceleration curves, peak velocity, world record limits
- **Baseball & softball:** pitch trajectories, launch angle optimization, ERA stabilization
- **Cycling & triathlon:** power output, total energy, GPS-computed distance
- **Basketball:** shot arc, free throw probability limits, scoring rate integrals

CHAPTER ONE

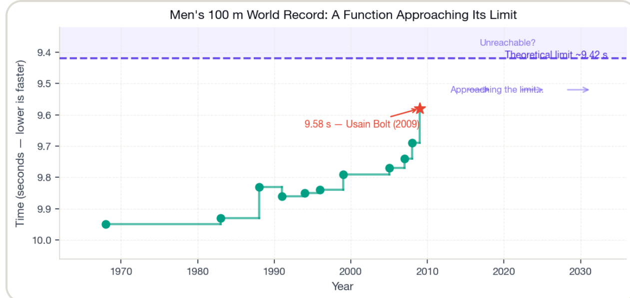
1 Limits

How close can you get to the boundary of what's possible?

The World Record That Can't Reach Zero

Since 1968, the men's 100 m world record has been broken 10 times — falling from 9.95 s all the way to Usain Bolt's 9.58 s in 2009. Every decade brings a new champion, a new fraction of a second shaved off.

But here's the question: *could the record eventually reach 9.0 seconds? 8.5? Zero?* Of course not. Biomechanics researchers estimate that the absolute physical minimum is approximately **9.4 seconds**. The records are *approaching* this boundary — getting closer and closer — but they can never reach it. This is exactly what mathematicians call a **limit**.



Each dot is an official world record. The dashed line is the theoretical biological minimum. Records approach this limit — never crossing it.

DEFINITION

$$\lim_{x \to a} f(x) = L$$

The limit of f(x) as x approaches a equals L. As x gets arbitrarily close to a, f(x) gets arbitrarily close to L — even if f(a) is undefined.

SPORTS TRANSLATION

$$\lim_{n \to \infty} WR(n) = 9.42 \text{ s}$$

As the number of records broken approaches infinity, the world record time approaches 9.42 s. Records get closer and closer, but never cross below this biological barrier.

Pro Tip: A limit describes what a function *approaches*, not necessarily what it *reaches*. A runner can approach perfect form all season without ever achieving it — that gap is the heart of limits.

MORE LIMITS IN SPORTS

- **Free throw %:** As a player shoots N free throws, their percentage approaches their true probability as $N \to \infty$
- **Pitcher ERA:** ERA stabilizes after ~200 batters faced. $\lim_{IP \to \infty}$ ERA(IP) = true talent level
- **Swim times:** Splits approach optimal pacing as training increases; physiology caps improvement

ONE-SIDED LIMITS

$$\lim_{x \to a^+} f(x) \qquad \lim_{x \to a^-} f(x)$$

Approaching from the right (+) or left (–). A two-sided limit exists only if both one-sided limits are equal.

Key Insight: A function can have a limit at a point even if it isn't defined there. The limit describes the *approach*, not the destination.

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CHAPTER TWO

Derivatives

What is your speed at exactly the 50-meter mark? Not your average — your instantaneous speed.

The Speedometer Problem

A sprinter's average speed over 100 m is easy: 100 ÷ 10.5 ≈ 9.52 m/s. But a coach wants to know: **how fast was the athlete moving at exactly t = 5 seconds?**

We estimate it with a *secant line* — average speed over a small interval around t = 5. The smaller the interval, the better the estimate. As the interval shrinks toward zero, the secant becomes the **tangent line**, whose slope is the **instantaneous velocity**. That slope is the derivative.

THE DERIVATIVE

$$f'(x) = \lim_{h\to 0} [f(x+h) - f(x)] / h$$

The derivative f′(x) is the instantaneous rate of change of f at x. It equals the slope of the tangent line to the graph of f at point (x, f(x)).

SPRINT PHYSICS

$$d(t) = v_{max}(t - \tau + \tau e^{-t/\tau})$$

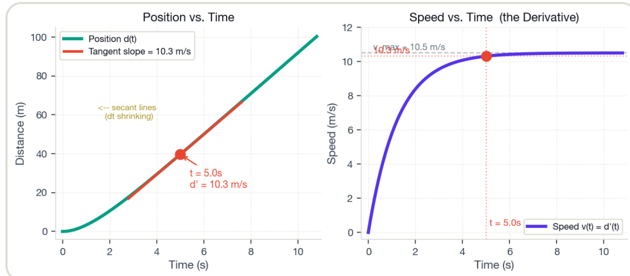
Furusawa sprint model. $v_{max} \approx 10.5$ m/s, τ = 1.25 s.

Its derivative: $v(t) = d'(t) = v_{max}(1 - e^{-t/\tau})$

KEY DERIVATIVE RULES IN SPORTS

- **Velocity:** v(t) = d′(t) — speed is the derivative of position
- **Acceleration:** a(t) = v′(t) = d″(t) — how quickly speed changes
- **Max height of a throw:** set h′(t) = 0 and solve — peak is where derivative = 0
- **Optimal launch angle:** d/dθ(Range) = 0 gives θ = 45° (in a vacuum)

Power Rule: If $d(t) = t^n$, then $d'(t) = n \cdot t^{n-1}$.
Baseball height: $h(t) = -16t^2 + 80t + 4$ ft → $h'(t) = -32t + 80$.
Peak: h′(t) = 0 → t = 2.5 s → h(2.5) = 104 ft.



Left: secant lines (dashed gold) converge to the tangent (red) as Δt → 0. Right: the velocity curve — the derivative of position.

AT T = 5 S

$$v(5) = 10.5(1 - e^{-5/1.25}) \approx 10.3 \text{ m/s}$$

The athlete is near peak velocity. The tangent slope on the position graph equals this value.

Chain Rule: If heart rate depends on speed, and speed changes over time:
dHR/dt = (dHR/dv) · (dv/dt)

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CHAPTER THREE

Integrals

Your GPS watch measures speed 10 times per second. How does it calculate total distance?

Adding Up the Moments

Suppose your GPS records your speed every second during a 10-second sprint. Each reading times 1 s gives a rough distance estimate. Add those 10 numbers and you have a **Riemann sum** — an approximation of total distance.

The more measurements per second, the more accurate the estimate. As the time intervals shrink toward zero, the sum becomes a **definite integral** — the exact total distance.

DEFINITE INTEGRAL

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \Sigma\, f(x_i)\, \Delta x$$

The definite integral equals the area under the curve of f from a to b. For a velocity function, it gives total distance traveled.

FUNDAMENTAL THEOREM OF CALCULUS

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

If $F' = f$, compute the exact integral by evaluating F at the endpoints. Derivatives and integrals are *inverses* of each other.

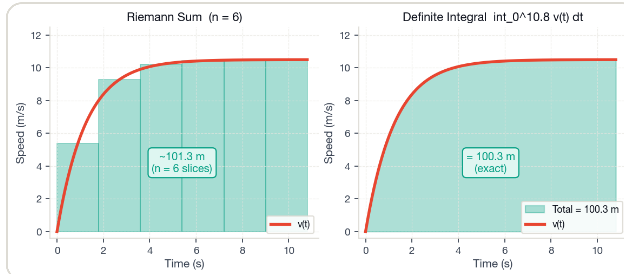
SPRINT EXAMPLE

$$\int_0^{10.8} v(t)\, dt = d(10.8) \approx 98 \text{ m}$$

Integrating the velocity function gives total distance. The antiderivative of $v(t) = v_{max}(1-e^{-t/\tau})$ is exactly d(t).

INTEGRALS EVERYWHERE IN SPORTS

- **GPS distance:** $\int v(t)\, dt$ — total ground covered in a match or race
- **Calories burned:** ∫(metabolic rate) dt — total energy expenditure over a game
- **Impulse:** $\int F(t)\, dt$ = change in momentum — force of a kick integrated over time
- **Swim volume:** ∫(yardage rate) dt — total yards across a practice set



Left: 6 rectangles approximate total distance. Right: the exact integral — area under the velocity curve.

Riemann Sum: With n = 6 rectangles, the estimate is ≈ 96 m. The exact integral gives ≈ 98 m. As $n \to \infty$, the Riemann sum converges to the integral — just like the definition of a limit!

The Big Connection:

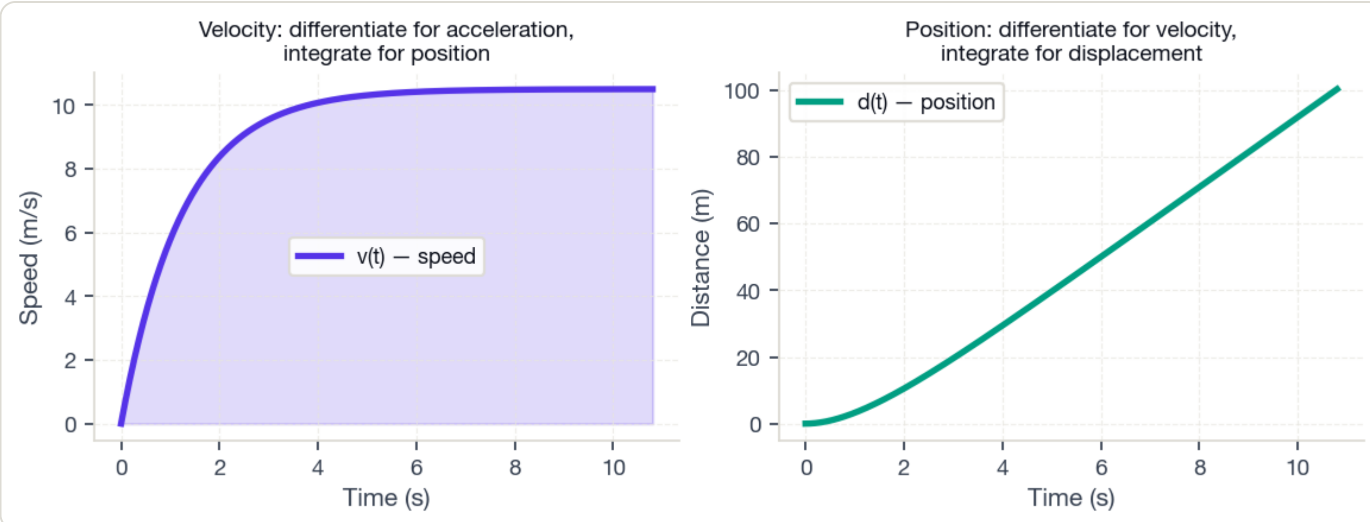
Position $\xrightarrow{d/dt}$ Velocity $\xrightarrow{d/dt}$ Acceleration

Acceleration $\xrightarrow{\int dt}$ Velocity $\xrightarrow{\int dt}$ Position

Differentiation and integration are *inverses* — the Fundamental Theorem of Calculus.

The Playbook

A quick reference + practice problems to sharpen your game.



The same sprint described two ways. Velocity (left) is the derivative of position. Position (right) is the integral of velocity. They contain identical information.

CONCEPT	QUESTION IT ANSWERS	FORMULA	SPORTS EXAMPLE
Limit	What value does a function approach?	$\lim_{x\to a} f(x) = L$	Biological minimum for the 100 m ≈ 9.42 s
Derivative	How fast is it changing right now?	$f'(x) = \lim_{h\to 0} [f(x+h) - f(x)]/h$	Instantaneous speed at the 50 m mark
Integral	How much has accumulated?	$\int_a^b f(x)\,dx = F(b) - F(a)$	Total distance from GPS velocity data

Practice Problems

Limits — The Closing Gap

A cyclist's average lap speed in successive races is: 24.8, 26.1, 27.2, 28.1, 28.8, 29.3, 29.7, 29.9… mph. Estimate $\lim_{n\to\infty}$ speed(n) and explain what it means physically.

Hint: The differences are 1.3, 1.1, 0.9, 0.7, 0.5, 0.4, 0.2 — decreasing. What value do the speeds converge to?

Derivatives — The Perfect Pitch

A baseball's height (feet) is $h(t) = -16t^2 + 64t + 4$. (a) Find $h'(t)$. (b) At what time does the ball reach its peak? (c) What is the maximum height?

Hint: Peak occurs when $h'(t) = 0$. Use the power rule: $d/dt(t^2) = 2t$.

Integrals — Marathon Pacing

A marathoner's velocity for the first 2 hours is $v(t) = 12 - 0.8t$ km/h (she slows with fatigue). Find $\int_0^2 v(t)\,dt$. How many kilometers does she cover?

Hint: Antiderivative of $(12 - 0.8t)$ is $12t - 0.4t^2$. Evaluate at $t = 2$ and $t = 0$.

All Three — Sprint Analysis

Using $v(t) = 10.5(1 - e^{-t/1.25})$: (a) What is $\lim_{t\to\infty} v(t)$? (b) Find $a(1) = v'(1)$. (c) Estimate $\int_0^5 v(t)\,dt$ with a 5-rectangle Riemann sum.

Hint for (a): as $t \to \infty$, $e^{-t/1.25} \to 0$, so $v \to 10.5$ m/s — the maximum speed.